

# Giant persistent current in a free-electron model with a flat Fermi surface

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**Abstract.** The tight-binding model is considered for a square lattice with filling factor  $1/2$ . The array has the shape of a rectangle with boundary conditions in both directions twisted by  $2\pi\phi_x$  and  $2\pi\phi_y$ . The components of the twist are associated with two components of the magnetic flux in torus geometry. An analytical expression is obtained for the energy and for the components of the persistent current (PC) at a given flux and temperature. It is shown that at zero temperature the PC density is proportional to the vector potential with a coefficient which does not depend on the size of the system. This happens because the Fermi surface for a square lattice at filling factor  $1/2$  is flat. Both the energy and the PC are periodic functions of the two flux components with the periods  $\phi_0/q$  and  $\phi_0/s$  where  $\phi_0 = hc/e$ , and  $q$  and  $s$  are integers which depend on the aspect ratio of the rectangle. The magnitude of the PC is the same as for superconductors. Therefore, a 3D system constructed from a macroscopic number of isolated coaxial cylinders at zero temperature is reminiscent of London's superconductor. It exhibits the quantization of trapped flux as well as the Meissner effect. However, all of the phenomena are of a mesoscopic nature. The critical field  $H_c$  decays with the effective size of the system,  $H_c \sim 1/R_{ef}$ . The magnitude of the PC decays with  $T$  as  $\exp(-\pi T R_{ef}/2at)$ , where  $t$  is the hopping amplitude and  $a$  is the lattice constant.

## 1. Introduction

Persistent current (PC) in mesoscopic structures [1, 2] has been extensively studied during the last decade both experimentally [3–5] and theoretically. The theoretical investigations concentrated on the role of different degrees of disorder [6, 7] and on the role of the interaction between electrons [8, 9].

The PC is a reaction of a system to an applied flux  $\Phi$ , or, equivalently, it can be described as a change of the energy of the system due to twisted boundary conditions. In a two-dimensional system which forms a cylinder, the twisted conditions mean that the wave function of a system acquires a factor  $\exp(i2\pi\Phi/\phi_0)$  upon a circulation of one electron around the axis of the cylinder. Here  $\phi_0 = hc/e$  is the flux quantum.

The flux  $\Phi$  is related to the tangential component  $A$  of the vector potential  $\Phi = 2\pi RA$ , where  $R$  is the radius of the cylinder. For a system with Galilean invariance the following simple statement is correct. The energy of a state with a given value of the tangential component  $P$  of the total momentum depends on  $A$  as

$$E(P, A) = E_0 - \frac{1}{2M} \left( P - \frac{Ne}{c} A \right)^2 \quad (1)$$

where  $N$  is the number of electrons and  $M = Nm$  is their total mass,  $m$  being the mass of one electron. The 2D current density  $j$  for a state with fixed  $P$  is

$$j_P = -\frac{c}{S} \left( \frac{\partial E}{\partial A} \right)_P = -\frac{ne^2}{mc} A + \frac{eP}{mS} \quad (2)$$

where  $S$  is the area of the cylinder surface and  $n = N/S$ . At  $P = 0$  equation (2) is reminiscent of the London equation for a superconducting current. In this case  $n$  should be the superfluid density.

A general derivation of equation (2), given above, is misleading because PC should be defined as a current in the ground state rather than in a state with fixed  $P$ . In two- or three-dimensional systems of free electrons the derivative of the energy with respect to  $A$  cannot be taken in such a simple way because the intervals of  $\Phi$ , where branches of the spectrum with different  $P$  change each other in the ground state, tend to zero with increasing system size.

For electrons in a periodic potential the situation is typically similar. The derivative of the energy with respect to the flux is large for a given branch. However, different branches replace each other in the ground state over such small intervals of  $\Phi$  that the derivative taken at a given total quasimomentum  $P$  does not reflect properties of the ground state. Scalapino *et al* [10] considered a tight-binding model with a two-dimensional square lattice. Their computations show that at filling factor  $\nu = 1/4$  the first level crossing occurs at  $\Phi \sim 1/L$ , where  $L$  is the size of the system. Their general conclusion is that the superfluid density, as found from the relation between  $j$  and  $A$ , is zero for free electrons in the tight-binding model. We show in this paper that this is not always the case.

We consider a 2D system of free electrons on a square lattice in a tight-binding approximation at filling factor  $\nu = 1/2$ . The shape of the system is assumed to be a rectangle with arbitrary aspect ratio. The specific features of the PC in this system have been demonstrated by Cheung *et al* [11] We show here that at  $T = 0$  the two-dimensional PC density does not depend on the size of the system and has a form

$$j = -\frac{4}{\pi^2} \frac{ne^2}{mc} (A - A_0). \quad (3)$$

Here  $m = \hbar^2/2ta^2$  is the electron mass,  $t$  being the nearest-neighbour hopping energy. The two-dimensional density is determined as  $n = 1/2a^2$ , where  $a$  is the lattice constant. For simplicity, we consider a system of spinless fermions. The generalization to the case of non-interacting fermions with spin is straightforward.

The constant  $A_0$  shows that the minimum of energy does not occur at zero flux. In contrast to equation (2), equation (3) describes PC in the ground state of the system which is a periodic function of  $\Phi$  with period  $\phi_0/q$ . Equation (3) is valid within the interval  $0 < \Phi < \phi_0/q$ , or  $0 < A < \phi_0/2\pi Rq$ , and is to be repeated periodically for other values of the flux. Here  $q$  is an integer which depends on the aspect ratio of the cylinder and on the type of the boundary conditions imposed in the direction of the cylinder axis. The first term is shown to be independent of the aspect ratio. Thus we have found that the PC density has the order of magnitude of the London current in superconductors.

Considering the 3D system constructed of a large number of coaxial closely packed 2D cylinders we show that it mimics the Meissner effect and the quantization of flux trapped in the opening.

These properties appear since the Fermi surface at  $\nu = 1/2$  is flat and no branch crossings occur over large intervals of  $\Phi$ . Say, for a square array, no branch crossing occurs over the whole interval  $0 < \Phi < \phi_0$ , which means that  $q = 1$ .

In fact, we are discussing a mesoscopic effect. The expression (3) is valid only at mesoscopically small temperature, and the ideal diamagnetism occurs for mesoscopically small values of magnetic field:

$$T < T_c \sim \frac{a}{R_{ef}} t \quad (4)$$

$$H < H_c \sim \frac{a}{R_{ef}} \sqrt{\frac{t}{a^2 b}} \quad (5)$$

where  $b$  is the spacing between neighbouring coaxial cylinders. The effective size  $R_{ef}$  is given by

$$R_{ef} = sD = q2\pi R = \sqrt{sq2\pi RD} \quad (6)$$

where  $D$  is the length of the cylinders and  $s$  and  $q$  are integers determined by the aspect ratio  $2\pi R/D$  (see below). Thus, this system can be classified as a ‘mesoscopic superconductor.’

Note that the average distance between energy levels in a 2D system is proportional to  $1/R^2$ . The  $1/R$  behaviour in the above equations is also a result of the flat Fermi surface at  $\nu = 1/2$ . As is seen from the calculations, all relevant interlevel distances are of the order of  $(a/R)t$ , rather than  $(a/R)^2 t$ .

Since both  $T_c$  and  $H_c$  vanish at large  $R$ , there are no real critical phenomena in this model system.

## 2. Calculation of PC at zero temperature

Consider a rectangle of  $L_x \times L_y$  lattice sites with periodic boundary conditions twisted in both directions by  $2\pi\Phi_x/\phi_0$  and  $2\pi\Phi_y/\phi_0$ . This corresponds to a toroidal geometry where  $\Phi_x$  is the flux through the cross section of the torus and  $\Phi_y$  is the flux through the opening.

The single-electron energies have the form

$$\epsilon(n_x, n_y, \phi_x, \phi_y) = -2t \left\{ \cos \left[ \frac{2\pi}{L_x} (n_x - \phi_x) \right] + \cos \left[ \frac{2\pi}{L_y} (n_y - \phi_y) \right] \right\} \quad (7)$$

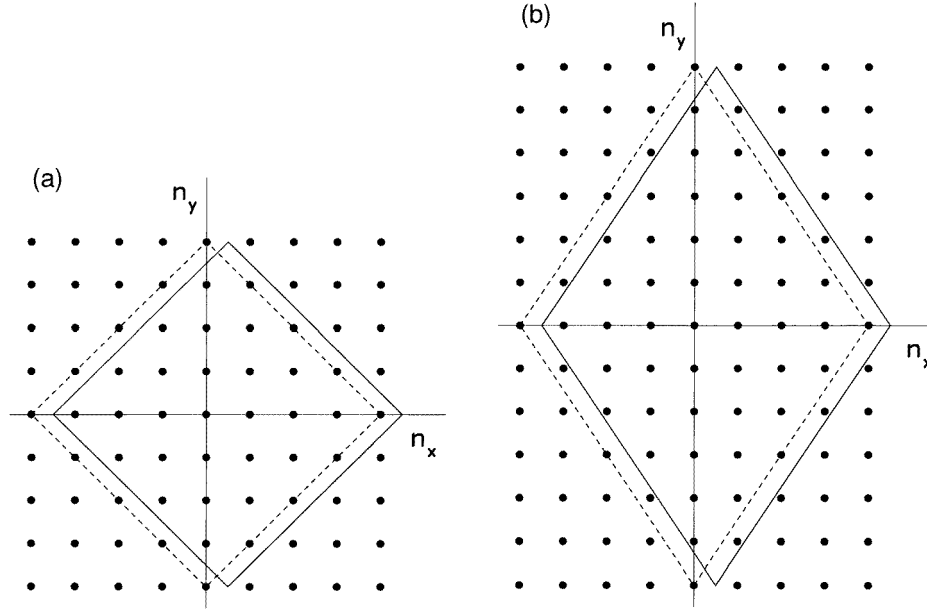
where we introduce dimensionless  $\phi_{x,y} = \Phi_{x,y}/\phi_0$  to simplify the notation. The values of integer quantum numbers  $n_x$  and  $n_y$  are restricted to the rectangle  $|n_{x,y}| \leq L_{x,y}/2$  (the first Brillouin zone). To find the energy of the ground state one has to sum  $\epsilon(n_x, n_y)$  over the values  $\{(n_x, n_y)\}$  inside the Fermi surface.

In principle, the calculation of PC can be performed either for constant number of particles  $N$  or for a constant value of chemical potential  $\mu$ . Generally speaking, these two definitions are not equivalent. It is important to note that such a problem does not exist at  $\nu = 1/2$  at even values of  $L_x, L_y$  at least. As one can see from the equation (7), every single-electron energy changes sign under the transformation  $n_x, n_y \rightarrow n_x + L_x/2, n_y + L_y/2$ . It follows that at  $\nu = 1/2$ , due to the electron–hole symmetry, the chemical potential  $\mu$  is zero at any value of the flux and at any temperature. Thus, if the flux changes at  $\mu = 0$ , the number of particles in the ground state of the system does not change and if the flux changes at a given number of particles such that  $\nu = 1/2$ , the chemical potential does not change.

Let us define the Fermi ‘surface’ (FS) in two-dimensional  $n_x, n_y$  space by the equation

$$\epsilon(n_x, n_y, \phi_x, \phi_y) = 0 \quad (8)$$

considering  $n_x, n_y$  as continuous variables. It is easy to see that the FS forms a rhomb at any value of the flux. Change in the flux produces a shift of the FS as a whole without changing its shape.



**Figure 1.** The Fermi surface at  $\nu = 1/2$ . The points represent allowed integer values of  $n_x$  and  $n_y$  inside the first Brillouin zone. The dashed lines show the Fermi surface at zero flux. The solid lines are the Fermi surface shifted by flux. The aspect ratio  $L_x/L_y = 1$  (a) and  $L_x/L_y = 2/3$  (b).

First, let us consider for simplicity a square sample,  $L_x = L_y$ . The FS forms a square shown in figure 1(a). At  $\phi_{x,y} = 0$  some of the allowed single-electron states lie exactly at the sides of this square. All of the states inside the square and half of the states at the sides of the square are occupied. All of the states at the sides have the same energy so the occupation numbers of these states are not defined, while the many-electron ground state is degenerate.

The degeneracy is lifted at infinitesimally small values of  $\phi$ . Suppose that  $\phi_y = 0$  and  $\phi_x > 0$ . Then the FS is shifted to the right (see figure 1(a)). All occupation numbers become defined. That is, the states at the right-hand side of the initial square get occupied and those at the left-hand side become empty. Note that the occupation numbers of as many as  $2L$  states change when  $\phi_x$  crosses zero.

It is easy to see that the occupation numbers are constant throughout the interval  $0 < \phi_x < 1$ . The total energy decreases with  $\phi_x$  and then increases again. At  $\phi_x = 1$  all electrons jump one step to the right and the Fermi surface restores its original position with respect to the lattice of integer numbers  $(n_x, n_y)$ . The total energy thus returns to the same value as at  $\phi_x = 0$ .

It follows that the total quasimomentum of the electron system in the ground state does not change through all of this interval and no branch crossing occurs. Thus the sum over the occupied states can be easily evaluated:

$$E(\phi_x, \phi_y) = \sum_{n_y=1}^{L/2-1} \sum_{n_x=-L/2+n_y+1}^{L/2-n_y} [\epsilon(n_x, n_y) + \epsilon(n_x, -n_y)] + \sum_{n_x=-L/2+1}^{L/2} \epsilon(n_x, 0)$$

$$= 8t \operatorname{Re} \frac{e^{2\pi i/L}}{(e^{2\pi i/L} - 1)^2} [(1 + e^{2\pi i/L})e^{-i2\pi\phi_x/L} + 2e^{-i2\pi\phi_y/L}]. \quad (9)$$

This expression is exact in the region  $0 < \phi_x \pm \phi_y < 1$ . In the limit of large  $L$ , the  $\phi$ -dependent part of the energy,  $\delta E(\phi_x, \phi_y) = E(\phi_x, \phi_y) - E(0, 0)$ , can be written in the form

$$\delta E(\phi_x, \phi_y) = 8t [\phi_y^2 - \phi_x(1 - \phi_x)]. \quad (10)$$

Repeating equation (10) periodically one gets the expression valid over the whole plane  $(\phi_x, \phi_y)$ :

$$\delta E(\phi_x, \phi_y) = 4t \left[ \left( \{\phi_+\} - \frac{1}{2} \right)^2 + \left( \{\phi_-\} - \frac{1}{2} \right)^2 - \frac{1}{2} \right] \quad (11)$$

where  $\phi_{\pm} = \phi_x \pm \phi_y$ , and  $\{\cdot\}$  denotes the fractional part of the number, defined as the difference between the number and the largest integer less than it.

Figure 2 shows the energy  $\delta E(\phi_x, \phi_y)$  as given by equation (11). The positions of the energy minima form a square lattice shifted from the origin:

$$(\phi_x, \phi_y) = \left( \frac{1+i+j}{2}, \frac{i-j}{2} \right) \quad (12)$$

with arbitrary integers  $i$  and  $j$ .

The point  $\phi_x = \phi_y = 0$  corresponds to a maximum of energy, in the same way as in 1D case with an even number of electrons. At this point the derivatives  $d\delta E/d\phi_{x,y}$  are discontinuous. Such behaviour appears as a result of lifting of the degeneracy of the states at the Fermi surface.

The PC at  $T = 0$  can be found as the derivative of the total energy with respect to the flux:

$$I_{x,y} = -c \frac{\partial E}{\partial \Phi_{x,y}} = -\frac{c}{\phi_0} \left( \frac{\partial E}{\partial \phi_+} \pm \frac{\partial E}{\partial \phi_-} \right) = -8 \frac{ct}{\phi_0} \left[ \left( \{\phi_+\} - \frac{1}{2} \right) \pm \left( \{\phi_-\} - \frac{1}{2} \right) \right]. \quad (13)$$

The magnitude of  $\delta E(\phi_x, \phi_y)$  and  $I_{x,y}(\phi_x, \phi_y)$  as given by equations (11) and (13) is independent of the size  $L$  of the square. Such a large magnitude results from the fact that in the region with no branch crossings (or, with no electron changing its state) all electrons together contribute to the current.

It may seem that the aspect ratio  $L_x/L_y = 1$  is crucial for the effect. In the next section we calculate PC at finite temperature for arbitrary aspect ratio  $L_x = sK$ ,  $L_y = qK$  with mutually prime integers  $s$  and  $q$ . We assume the macroscopic limit  $K \rightarrow \infty$ . It is useful to generalize the definition of  $\phi_{\pm}$  for a rectangular sample as follows:

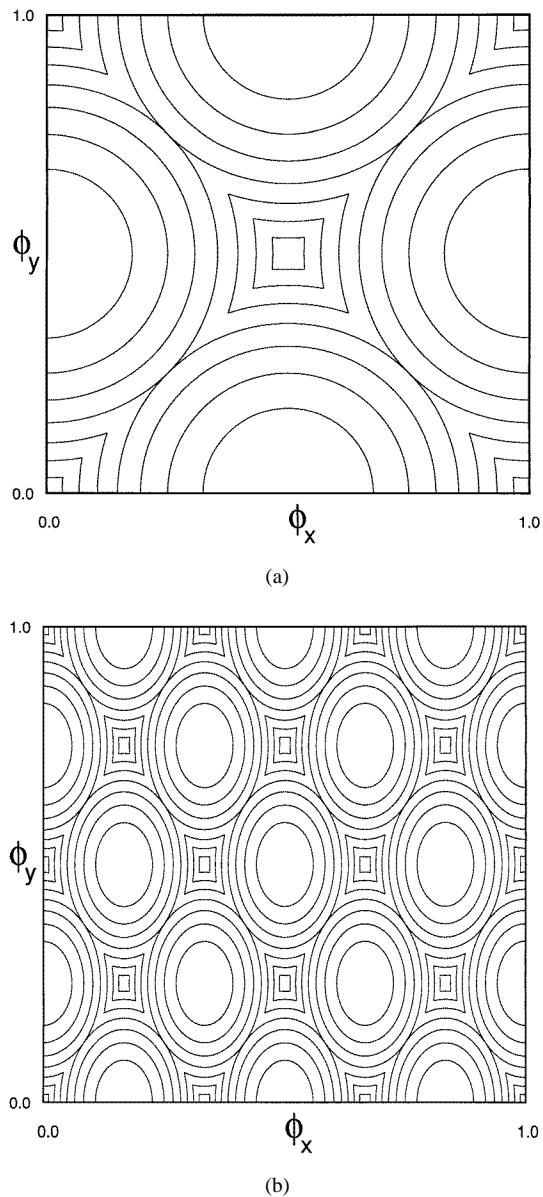
$$\phi_{\pm} = q\phi_x \pm s\phi_y. \quad (14)$$

With this notation our results for the limit  $T = 0$  read

$$\begin{aligned} I_x &= -\frac{8}{sq} \frac{ct}{\phi_0/q} \left[ \left( \{\phi_+\} - \frac{1}{2} \right) + \left( \{\phi_-\} - \frac{1}{2} \right) \right] \\ I_y &= -\frac{8}{sq} \frac{ct}{\phi_0/s} \left[ \left( \{\phi_+\} - \frac{1}{2} \right) - \left( \{\phi_-\} - \frac{1}{2} \right) \right]. \end{aligned} \quad (15)$$

The flux-dependent part of the energy can be restored from equation (15):

$$\delta E(\phi_x, \phi_y) = \frac{4t}{sq} \left[ \left( \{\phi_+\} - \frac{1}{2} \right)^2 + \left( \{\phi_-\} - \frac{1}{2} \right)^2 - \frac{1}{2} \right]. \quad (16)$$



**Figure 2.** Lines of constant  $\delta E(\phi_x, \phi_y)$  as given by equation (16) for a square sample with  $s = q = 1$  (a), and for a rectangle with  $s = 2$  and  $q = 3$  (b). Note the difference in periodicity.

This result is a generalization of equation (11) to an arbitrary aspect ratio  $s/q$  of the rectangular sample.

As follows from equations (15), (16), the energy and current as functions of flux do depend on the aspect ratio. However, they do not depend on the system size, if the aspect ratio is kept constant. The result, equations (15), (16), can be understood from figure 1(b), which is drawn for the case where  $L_x/L_y = 2/3$ . Unlike the case for figure 1(a), there are now points  $(n_x, n_y)$  closer to the Fermi surface than one lattice spacing. However, there

is still a regularity in their positions. That is, as the Fermi surface shifts with the flux, the points enter the Fermi sea in groups. Consider, for example, the same case as above:  $\phi_y = 0$  and  $\phi_x > 0$ . As seen in figure 1(b), the branch crossings occur only at  $\phi_x = 2\pi/3$ ,  $4\pi/3$  and  $2\pi$ . In terms of  $\phi_{\pm}$  this corresponds to integer  $\phi_{\pm} = 3\phi_x = 1, 2$  and  $3$ . These values of the flux are determined by  $s$  and  $q$  and do not change with the size of the system. The number of points in each group is, in turn, proportional to the size of the system, so the corresponding contribution to the current is large.

At  $\Phi_y = 0$  equation (15) gives

$$I_x = -\frac{16}{sq} \frac{ct}{\phi_0/q} \left( \left\{ \frac{\Phi_x}{\phi_0/q} \right\} - \frac{1}{2} \right). \quad (17)$$

A similar equation has been obtained in reference [11] for the cylinder geometry. The difference from the torus geometry considered here is in the boundary conditions along the cylinder axis, chosen as  $y$ . In what follows we assume periodic boundary conditions in this direction with  $\Phi_y = 0$ . This leads to equation (17) for a total current through the cylinder. Cheung *et al* [11] impose the condition that the wave function is zero at the edges of the cylinder. Their result is formulated in terms of a Fourier series. It can be shown that for their boundary conditions the second term in equation (17) changes while the first term remains intact. Note that both energy and current are periodic functions of flux with period  $\phi_0/q$  rather than  $\phi_0$ .

Taking into account that the current density  $j_x = I_x/(aL_y)$  and that the vector potential  $A_x = \Phi_x/(aL_x)$  one obtains equation (3) with the first term independent of  $s$  and  $q$ .

### 3. PC at finite temperature

We start with the equation

$$I_x = -\frac{c}{\phi_0} \sum_{n_x=0}^{sK-1} \sum_{n_y=0}^{qK-1} \frac{\partial \epsilon(n_x, n_y)}{\partial \phi_x} \frac{1}{1 + \exp(\epsilon(n_x, n_y)/T)}. \quad (18)$$

It is convenient to rewrite the single-electron energy in the form

$$\epsilon(n_x, n_y) = -4t \cos\left(\frac{\pi}{sqK}(n_+ - \phi_+)\right) \cos\left(\frac{\pi}{sqK}(n_- - \phi_-)\right) \quad (19)$$

where  $n_{\pm} = qn_x \pm sn_y$  and  $\phi_{\pm}$  are given by equation (14). Using  $\partial/\partial\phi_x = q(\partial/\partial\phi_+ + \partial/\partial\phi_-)$  we find that the current has two terms:

$$I_x = \frac{1}{s}(I_+ + I_-) \quad (20)$$

where

$$I_{\pm} = -sq \frac{c}{\phi_0} \sum_{n_x=0}^{sK-1} \sum_{n_y=0}^{qK-1} \frac{\partial \epsilon(n_x, n_y)}{\partial \phi_{\pm}} \frac{1}{1 + \exp(\epsilon(n_x, n_y)/T)}. \quad (21)$$

The idea of our calculation is to transform equation (21) in such a way that the internal sum gives the PC of the 1D problem with an effective temperature and effective flux. For this purpose we use the identity

$$\sum_{n_x=0}^{sK-1} \sum_{n_y=0}^{qK-1} f(n_x, n_y) = \sum_{m=0}^{sK-1} \sum_{d=0}^{q-1} \sum_{k=0}^{K-1} f(m+d+sk, d+qk). \quad (22)$$

This identity is valid for any function  $f(n_x, n_y)$  periodic in  $n_x$  and  $n_y$  with periods  $sK$  and  $qK$  respectively. Then  $I_+$  can be written in the form

$$I_+ = -\frac{4\pi ct}{K\phi_0} \sum_{m=0}^{sK-1} \sum_{d=0}^{q-1} \sum_{k=0}^{K-1} \left( \left[ \sin\left(\frac{\pi}{sqK}(n_+ - \phi_+)\right) \cos\left(\frac{\pi}{sqK}(n_- - \phi_-)\right) \right] \right. \\ \left. \times \left\{ 1 + \exp\left[-\frac{4t}{T} \cos\left(\frac{\pi}{sqK}(n_+ - \phi_+)\right) \cos\left(\frac{\pi}{sqK}(n_- - \phi_-)\right)\right] \right\}^{-1} \right). \quad (23)$$

A similar expression can be written out for  $I_-$ . Note that  $n_- = qm + (q-s)d$  does not depend on  $k$ , while  $n_+ = (qm + qd + sd) + 2sqk$  does depend on  $k$ . Therefore, the current  $I_+$  can be written as

$$I_+ = \sum_{m=0}^{sK-1} \sum_{d=0}^{q-1} \mathcal{I}_+(m, d) \quad (24)$$

where  $\mathcal{I}_+(m, d)$  denotes the internal sum over  $k$ ,

$$\mathcal{I}_+(m, d) = -\frac{4\pi ct}{K\phi_0} \frac{2T}{\tilde{T}} \sum_{k=0}^{K-1} \left( \left[ \sin\left(\frac{2\pi}{K}(k - \tilde{\phi}_+)\right) \right] \right. \\ \left. \times \left\{ 1 + \exp\left[-\frac{2t}{\tilde{T}} \cos\left(\frac{2\pi}{K}(k - \tilde{\phi}_+)\right)\right] \right\}^{-1} \right). \quad (25)$$

The sum in equation (25) describes the PC in a 1D system with an effective temperature and effective flux given by

$$\tilde{T}(m, d) = T \left/ \left[ 2 \cos\left(\frac{\pi}{sqK}(qm + (q-s)d - \phi_-)\right) \right] \right. \\ \tilde{\phi}_+(m, d) = \frac{\phi_+ - qm - qd - sd}{2sq}. \quad (26)$$

Using the Poisson summation formula (see reference [12]), one obtains

$$\mathcal{I}_+(m, d) = \frac{8\pi cT}{\phi_0} \sum_{l=1}^{\infty} \frac{\cos(l\pi K/2)}{\sinh(l\pi \tilde{T} K/2t)} \sin(2\pi l \tilde{\phi}_+). \quad (27)$$

Performing the summation over  $m$  and  $d$  in equation (24) we note that  $\tilde{T}$  is a smooth function of  $m/K$  and  $d/K$ . However,  $\sin(2\pi l \tilde{\phi}_+)$  has an oscillatory behaviour for some  $l$ , so the contribution of the corresponding harmonics vanishes in the limit  $K \rightarrow \infty$ . The oscillatory behaviour is absent if  $l$  is an integer multiple of  $2sq$ . For these  $l$ , the sum over  $m$  can be transformed into an integral via  $p = (\pi/sK)m$ , while the sum over  $d$  simply gives a factor  $q$ . Thus, one obtains

$$I_{\pm} = \sum_{l=1}^{\infty} A_l \sin(2\pi l \phi_{\pm}) \quad (28)$$

where

$$A_l = sqKT \frac{2c}{\phi_0} \int_0^{\pi} \frac{dp}{\sinh(l\pi sqKT/2t \sin p)}. \quad (29)$$



For the PC in the  $x$ -direction one has from equation (20)

$$I_x = \frac{1}{s} \sum_{l=1}^{\infty} A_l [\sin(2\pi l \phi_+) + \sin(2\pi l \phi_-)]. \quad (30)$$

A similar calculation gives

$$I_y = \frac{1}{q} \sum_{l=1}^{\infty} A_l [\sin(2\pi l \phi_+) - \sin(2\pi l \phi_-)]. \quad (31)$$

Equations (29) and (30) give the Fourier series expansion of PC at any temperature. Expansion of  $A_l$  at small  $KT/t$  yields

$$A_l \approx \frac{8t}{\phi_0} \frac{1}{l\pi}. \quad (32)$$

In this case equations (30), (31) and (32) give the Fourier series expansion of the zero-temperature result, equation (15).

In the opposite limit,  $KT/t \gg 1$ , the amplitudes of the harmonics decay as

$$A_l \approx \frac{8c}{\phi_0 \sqrt{l}} \sqrt{sqKTt} \exp\left(-\frac{l\pi sqKT}{2t}\right) \quad (33)$$

so the PC is dominated by its lowest harmonic. When  $\Phi_y = 0$  one has

$$I_x \approx \frac{1}{sq} \frac{16c}{\phi_0/q} \left(\frac{R_{ef}Tt}{a}\right)^{1/2} \exp\left(-\frac{\pi R_{ef}T}{2at}\right) \sin\left(\frac{\Phi_x}{\phi_0/q}\right). \quad (34)$$

A similar result was obtained in reference [11].

#### 4. Low-temperature magnetic properties

In this section we study the magnetic properties of a quasi-3D system constructed of a macroscopic number of closely packed coaxial cylinders assuming that the temperature is very low. Thus the connection between flux and current for each cylinder is given by equation (17). For the sake of simplicity we assume that the cylinders are long, such that the circumference of the outer cylinder  $2\pi R = aL_x$  is much smaller than the length  $D = aL_y$ .

The distance between the internal and external cylinders is supposed to be much less than  $R$ . We assume further that all of the cylinders have the same ratio  $L_x/L_y = s/q$ . One can imagine a small change either in  $L_x$  and  $L_y$  of adjacent cylinders or in their lattice constant.

The second term in equation (17) appears since zero flux does not correspond to the minimum of energy. It may lead to the appearance of a spontaneous flux in this system. This idea has been put forward by Wohlleben *et al* and Szopa and Zipper, reference [13], and then studied in detail in reference [14]. These authors considered a cylinder constructed from isolated 1D rings. Loss and Martin [15] argued that in a single 1D ring no symmetry breaking can occur, but their arguments are restricted to the 1D case.

In this paper we concentrate on the first term in equation (17). It is an analogue of the London current in superconductors and it creates a strong diamagnetism in the quasi-3D system described above. Suppose that an external magnetic field  $H_{ext}$  is applied to the system and that there is a solenoid creating flux  $\Phi_{ext}$  inside the internal cylinder.

Let  $\Phi_k$  be the total flux inside cylinder  $k$ , where  $k = 1$  for the internal cylinder and  $k = N$  for the external one. The flux obeys the equation

$$\Phi_k - \Phi_{k-1} = 2\pi Rb \left( H_{ext} + \frac{4\pi}{cD} \sum_{i=k}^N I(\Phi_i) \right). \quad (35)$$

Here  $b$  is the distance between adjacent cylinders which we assume to be of the order of the lattice constant  $a$ . Since the thickness  $d = Nb$  is supposed to be much less than  $R$  we have neglected that the radii of cylinders are slightly different. The right-hand side of equation (35) describes the flux through the area between the  $k$ th and  $(k - 1)$ th cylinders created by the external field and outer cylinders.

The following condition should be added to this finite-difference equation:

$$\Phi_1 - \Phi_{ext} = \pi R^2 \left( H_{ext} + \frac{4\pi}{cD} \sum_{i=1}^N I(\Phi_i) \right). \quad (36)$$

If  $\Phi_k$  is a smooth function of  $k$  one can transform equation (35) into a differential equation:

$$\frac{d^2\Phi}{dr^2} = \frac{\phi_0/q}{\lambda^2} \left( \left\{ \frac{\Phi}{\phi_0/q} \right\} - \frac{1}{2} \right). \quad (37)$$

Here  $\lambda$  is the analogue of the London penetration depth:

$$\lambda^{-2} = \frac{4\pi}{b} \frac{16t}{\phi_0^2} = \frac{16e^2 n_3}{\pi m c^2} \quad (38)$$

where  $n_3 = 1/2ba^2$  is the 3D electron density.

Equation (36) transforms into the boundary condition at  $r = R$ :

$$\Phi(R) - \Phi_{ext} = \frac{R}{2} \frac{d\Phi}{dr} \Big|_{r=R}. \quad (39)$$

The second boundary condition reads

$$\frac{d\Phi}{dr} \Big|_{r=R+d} = 2\pi R H_{ext}. \quad (40)$$

One can use the differential equation if  $\lambda \gg b$ .

Equation (37) can also be obtained by minimizing total energy with respect to flux. The total energy consists of two parts. The first is the energy of the magnetic field in the space between the cylinders. The magnetic field can be expressed through  $d\Phi/dr$  using equation (35) as follows:

$$\frac{d\Phi}{dr} = 2\pi R H(r). \quad (41)$$

The second part is the internal energy of a 2D electron gas. This energy per cylinder is given by equation (16) at  $\phi_y = 0$ . Thus, one gets for the total energy

$$E_{total} = \frac{1}{8\pi} \frac{D}{2\pi R} \int \left( \frac{d\Phi}{dr} \right)^2 dr + \int \delta E(\Phi) \frac{dr}{b}. \quad (42)$$

Minimizing this expression with respect to  $\Phi(r)$  and taking into account that  $d\delta E/d\Phi = (-1/c)I(\Phi)$ , where  $I(\Phi)$  is given by equation (17), one obtains equation (37).

Equation (37) is non-linear since it contains the fractional part  $\Phi/(\phi_0/q)$  which makes the right-hand side periodic. However, it becomes linear if the total drop of the flux inside the system is smaller than  $\phi_0/q$ . If  $H_{ext} = 0$  the solution of the linearized equation with boundary conditions (39), (40) in the case where  $R \gg d \gg \lambda$  is

$$\Phi(r) = \Phi_n + (\Phi_{ext} - \Phi_n) \frac{2\lambda}{R} \exp\left(-\frac{r-R}{\lambda}\right). \quad (43)$$

Here

$$\Phi_n = \frac{\phi_0}{q} \left( n - \frac{1}{2} \right). \quad (44)$$

One can see that the flux inside the cylinder with  $d \gg \lambda$  may take only quantized values  $\Phi_n$  with arbitrary integer  $n$ . Note that there is no zero flux among the allowed values of the frozen flux  $\Phi_n$ . This is because zero flux does not correspond to a minimum of the total energy at zero temperature. The solution, equation (43), is obtained in the linear approximation and it is valid if  $(\Phi_{ext} - \Phi_n)2\lambda/R < \phi_0/q$ . The interpretation of this result is that the inner cylinders carry a current which creates a favourable flux for the rest of the system.

If the system is in an external magnetic field  $H_{ext}$ , the solution is

$$\Phi(r) = \Phi_n + 2\pi R\lambda H_{ext} \exp\left(-\frac{R+d-r}{\lambda}\right) \quad (45)$$

or in terms of the magnetic field defined by equation (41)

$$H(r) = H_{ext} \exp\left(-\frac{R+d-r}{\lambda}\right). \quad (46)$$

In this case the cylinders near the external surface carry current which screens the magnetic field inside the system and adjusts the total flux to  $\Phi_n$ . The solution is valid if  $2\pi R\lambda H_{ext} < \phi_0/q$ . This condition is equivalent to equation (5). It has a simple interpretation. The loss in the total energy due to the ideal Meissner effect is of the order of  $H_{ext}^2 R D b$  per cylinder. The gain in the energy of a cylinder due to the adjusted flux is of the order of  $t/sq$  (see equation (16)). At large field the loss becomes larger than the gain and the field penetrates into the system. This is the origin of a ‘mesoscopic’ critical field. Note that the relation  $H_c^2 R D b \sim t/sq$  is also equivalent to equation (5).

It follows from the results of the previous section that the zero-temperature approximation holds if  $sqKT/t = T\sqrt{sq2\pi RD}/at \ll 1$ . This is the same condition as equation (4). At larger temperatures the penetration depth  $\lambda$  increases as  $\exp(\pi T R_{ef}/4at)$  and eventually reaches the thickness  $d$  of the cylinder, gradually destroying the strong diamagnetism.

## 5. Conclusions

We have presented a model which mimics on a mesoscopic scale some properties of superconductors, such as the Meissner effect and quantization of flux, though the physics of the model does not involve any electron pairing. The flux quantum in the model is  $\phi_0/q$  where  $q$  is determined by the aspect ratio of the system.

Since the range of temperature and magnetic field for these phenomena shrinks to zero in a macroscopic system, one should not expect any phase transitions. However, for a mesoscopic system this range is not necessarily small. Let us assume a hypothetical 3D layered system with very weak interaction between layers and a flat two-dimensional Fermi surface. Then it follows from equations (4), (5) that the temperature range is up to 12 K and the range of  $H_{ext}$  is up to 240 G for a system with  $R_{ef} = 3 \times 10^{-5}$  cm,  $a = b = 3 \times 10^{-8}$  cm and  $t = 1$  eV. In a system with disorder the obvious condition for these phenomena is that the elastic mean free path is smaller than the size  $R_{ef}$ .

Our model ignores electron–electron interaction. We hope that it is not important at large  $t$ . Our modelling of small interacting systems, up to 18 electrons, shows the same value of the PC at  $t$  immediately above the Wigner crystal quantum melting point [16].

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