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Analytic Coulomb matrix elements in the lowest Landau level in disk geometry

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Using Darling's theorem on products of generalized hypergeometric series, an analytic expression is obtained for the Coulomb matrix elements in the lowest Landau level in the representation of angular momentum. The result is important in the studies of fractional quantum Hall effect (FQHE) in disk geometry. Matrix elements are expressed as simple finite sums of positive terms, eliminating the need to approximate these quantities with slowly convergent series. As a by-product, an analytic representation for certain integrals of products of Laguerre polynomials is obtained. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1446244]

I. INTRODUCTION

The following integrals,

$$M_{mn}^{l}=(m+l,n|r_{12}^{-1}|m,n+l)=\int\int d^2r_1 d^2r_2\, \psi_{m+l}^{*}(\mathbf{r}_1)\psi_n^{*}(\mathbf{r}_2)\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\psi_m(\mathbf{r}_1)\psi_{n+l}(\mathbf{r}_2), \quad (1)$$

represent the Coulomb interaction matrix elements in the lowest Landau level. These are the basic quantities for studies of correlated two-dimensional systems in quantizing magnetic fields.[1–7] The single-particle wave functions in the angular momentum representation are given by

$$\psi_m(\mathbf{r})=(2\pi 2^m m!)^{-1/2} r^m e^{im\phi-r^2/4}, \quad (2)$$

where r and ϕ are polar coordinates in the plane, and the magnetic length $\sqrt{\hbar c/eH}$ is taken as a unit of length. The axial gauge for the vector potential $\mathbf{A}=\frac{1}{2}[\mathbf{H},\mathbf{r}]$ is chosen.

Full Coulomb interaction has been shown to play a crucial role in edge effects in fractional quantum Hall systems, not captured by Laughlin's wave function.[1,2] The results of Refs. 1 and 2 would have been difficult to obtain without an analytic formula for M_{mn}^{l}, the derivation of which is the subject of this work. Use of well-known expressions by Girvin and Jach[3] is prohibitive at moderately large m and n because of large cancellations. The problem was addressed in Ref. 8, where slowly convergent series to approximate M_{mn}^{l} have been derived.

Here we present analytic formulas for M_{mn}^{l} that contain simple finite sums of positive terms, which can be easily evaluated for any m, n, and l. Moreover, the symmetry with respect to interchanging m and n is explicitly preserved.

II. FORMULA FOR MATRIX ELEMENTS

We start with the result, which reads

$$M_{mn}^{l}=\sqrt{\frac{(m+l)!(n+l)!}{m!n!}}\frac{\Gamma(l+m+n+\frac{3}{2})}{\pi 2^{l+m+n+2}}[A_{mn}^{l}B_{nm}^{l}+B_{mn}^{l}A_{nm}^{l}], \quad (3)$$

where

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$$A^l_{mn}=\sum_{i=0}^{m}\binom{m}{i}\frac{\Gamma(\frac{1}{2}+i)\Gamma(\frac{1}{2}+l+i)}{(l+i)!\Gamma(\frac{3}{2}+l+n+i)}, \tag{4a}$$

and

$$B^l_{mn}=\sum_{i=0}^{m}\binom{m}{i}\frac{\Gamma(\frac{1}{2}+i)\Gamma(\frac{1}{2}+l+i)}{(l+i)!\Gamma(\frac{3}{2}+l+n+i)}(\tfrac{1}{2}+l+2i). \tag{4b}$$

The rest of the article presents the derivation of Eqs. (3) and (4). First, we substitute

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}=\int\frac{d^2q}{2\pi q}\exp[i\mathbf{q}(\mathbf{r}_1-\mathbf{r}_2)] \tag{5}$$

into Eq. (1). The two separate integrals over $\mathbf{r}_1$ and $\mathbf{r}_2$ can be evaluated in terms of Laguerre polynomials $L^l_m(q^2/2)$.[9] Substituting $q^2/2=x$ we obtain

$$M^l_{mn}=\sqrt{\frac{m!n!}{2(m+l)!(n+l)!}}\int dx x^{l-1/2}e^{-2x}L^l_m(x)L^l_n(x). \tag{6}$$

The above integral can be expressed[10] using generalized hypergeometric function[11,12]

$$M^l_{mn}=\sqrt{\frac{(l+m)!}{2\pi m!n!(l+n)!}}\frac{\Gamma(n+\frac{1}{2})\Gamma(l+\frac{1}{2})}{l!}F\left(\begin{matrix}1/2, & l+\frac{1}{2}, & l+m+1\\ & -n+\frac{1}{2}, & l+1\end{matrix}\middle| z\right) \tag{7}$$

in the limit $z\rightarrow -1$, approached from the right. The function F is defined as

$$F\left(\begin{matrix}a, & b, & c\\ & d, & e\end{matrix}\middle| z\right)=\sum_{i=0}^{\infty}\frac{z^i(a)_i(b)_i(c)_i}{i!(d)_i(e)_i}, \tag{8}$$

where $(z)_i=z(z+1)\cdots(z+i-1)=\Gamma(z+i)/\Gamma(z)$. Taking the limit avoids problems at $z=-1$, which is at the radius of convergence of the power series (8). For $|z|<1$ the right-hand side of Eq. (7) gives a more general integral of $x^{l-1/2}e^{(z-1)x}L^l_m(-zx)L^l_n(x)$, which is analytic in z.

When one of the upper parameters is a negative integer, the series (8) terminate yielding a finite sum. At $z=-1$ we have

$$F\left(\begin{matrix}-k, & b, & c\\ & d, & e\end{matrix}\middle| -1\right)=\sum_{i=0}^{k}\binom{k}{i}\frac{(b)_i(c)_i}{(d)_i(e)_i}. \tag{9}$$

Since none of the upper parameters of F in Eq. (7) are negative integers, the series is infinite. However, it appears possible to transform Eq. (7) in such a way that the result contains only terminating hypergeometric series. Using Darling's theorem on products,[13] the infinite series (7) are brought into a sum of products of generalized hypergeometric series that each have at least one negative integer upper argument, therefore, representing a finite sum. Darling's theorem for the function F reads

$$
\begin{aligned}
(1-z)^{a+b+c-d-e} F\left(\begin{matrix} a, & b, & c \\ & d, & e \end{matrix}\middle| z\right)
&= \frac{e-1}{e-d} F\left(\begin{matrix} d-a, & d-b, & d-c \\ & d, & d+1-e \end{matrix}\middle| z\right) F\left(\begin{matrix} e-a, & e-b, & e-c \\ & e-1, & e+1-d \end{matrix}\middle| z\right) \\
&\quad + \frac{d-1}{d-e} F\left(\begin{matrix} e-a, & e-b, & e-c \\ & e, & e+1-d \end{matrix}\middle| z\right) F\left(\begin{matrix} d-a, & d-b, & d-c \\ & d-1, & d+1-e \end{matrix}\middle| z\right). \qquad (10)
\end{aligned}
$$

Using Eq. (10) and setting $z=-1$ in the end we obtain

$$
\begin{aligned}
M^l_{mn} &= \sqrt{\frac{(l+m)!}{\pi m!n!(l+n)!}} \frac{\Gamma(n+\frac{1}{2})\Gamma(l+\frac{1}{2})}{2^{l+m+n+1}(l+n+\frac{1}{2})l!} \\
&\times \Bigg\{ (n+\tfrac{1}{2}) F\left(\begin{matrix} -m, & \frac{1}{2}, & l+\frac{1}{2} \\ & l+1, & l+n+\frac{3}{2} \end{matrix}\middle| -1\right) F\left(\begin{matrix} -n, & -n-l, & -n-l-m-\frac{1}{2} \\ & -n-\frac{1}{2}, & -n-l+\frac{1}{2} \end{matrix}\middle| -1\right) \\
&+ l F\left(\begin{matrix} -m, & \frac{1}{2}, & l+\frac{1}{2} \\ & l, & l+n+\frac{3}{2} \end{matrix}\middle| -1\right) F\left(\begin{matrix} -n, & -n-l, & -n-l-m-\frac{1}{2} \\ & -n+\frac{1}{2}, & -n-l+\frac{1}{2} \end{matrix}\middle| -1\right) \Bigg\}. \qquad (11)
\end{aligned}
$$

Further, we prove the following hypergeometric identity, valid for any positive integer k:

$$
F\left(\begin{matrix} -k, & -k-a, & -k-b \\ & -k-c, & -k-d \end{matrix}\middle| -1\right) = \frac{(1+a)_k(1+b)_k}{(1+c)_k(1+d)_k} F\left(\begin{matrix} -k, & 1+c, & 1+d \\ & 1+a, & 1+b \end{matrix}\middle| -1\right). \qquad (12)
$$

The proof is obtained by reversing the order of summation in the (finite) sum, $i \to k-i$, using the symmetry of the binomial coefficients with respect to this substitution, and noticing that $(-k-a)_{k-i} = (-1)^{k-i}(1+a)_k/(1+a)_i$. Using identity (12) to transform the second hypergeometric function in each of the two terms in Eq. (11), we get

$$
\begin{aligned}
M^l_{mn} &= \sqrt{\frac{(l+m)!(l+n)!}{m!n!}} \frac{\Gamma(l+m+n+\frac{3}{2})}{\pi 2^{l+m+n+1}(l+n+\frac{1}{2})} \\
&\times \Bigg\{ \sum_{i=0}^{m} \binom{m}{i} \frac{\Gamma(\frac{1}{2}+i)\Gamma(\frac{1}{2}+l+i)}{(l+i-1)!\Gamma(\frac{3}{2}+l+n+i)} \sum_{j=0}^{n} \binom{n}{j} \frac{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+l+j)}{(l+j)!\Gamma(\frac{3}{2}+l+m+j)} \\
&+ \sum_{i=0}^{m} \binom{m}{i} \frac{\Gamma(\frac{1}{2}+i)\Gamma(\frac{1}{2}+l+i)}{(l+i)!\Gamma(\frac{3}{2}+l+n+i)} \sum_{j=0}^{n} \binom{n}{j} \frac{\Gamma(\frac{3}{2}+j)\Gamma(\frac{1}{2}+l+j)}{(l+j)!\Gamma(\frac{3}{2}+l+m+j)} \Bigg\}. \qquad (13)
\end{aligned}
$$

The two terms in Eq. (13) can be brought together, restoring the symmetry with respect to m and n:

$$
\begin{aligned}
M^l_{mn} &= \sqrt{\frac{(l+m)!(l+n)!}{m!n!}} \frac{\Gamma(l+m+n+\frac{3}{2})}{\pi 2^{l+m+n+1}(l+n+\frac{1}{2})} \sum_{i=0}^{m} \sum_{j=0}^{n} (l+i+j+\tfrac{1}{2}) \\
&\times \binom{m}{i} \frac{\Gamma(\frac{1}{2}+i)\Gamma(\frac{1}{2}+l+i)}{(l+i)!\Gamma(\frac{3}{2}+l+n+i)} \binom{n}{j} \frac{\Gamma(\frac{1}{2}+j)\Gamma(\frac{1}{2}+l+j)}{(l+j)!\Gamma(\frac{3}{2}+l+m+j)}. \qquad (14)
\end{aligned}
$$

Finally, regrouping the terms to split the double sum back into a product of single sums while preserving the $m \leftrightarrow n$ symmetry we arrive at Eqs. (3) and (4).

We note that Eqs. (3) and (4) also represent a useful analytic representation for the integrals of products of Laguerre polynomials (6).

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